<u>Continuity Equation</u>

The fundamental principle expressed by the Continuity Equation is that the mass of the atmosphere is constant. Circulation patterns can move that mass around, but the total mass stays basically the same over reasonable time spans. For air parcels of constant volume, the mass is just the density. Viewing things this way allows us to use the principle of continuity to see how density changes can occur in our atmosphere given reasonable constraints.

Imagine an air column that extends through the troposphere. One way we can change the mass of air in that air column is by importing air streams into that air column with a different density than the air streams leaving the air column. Consider the example in Figure 1.

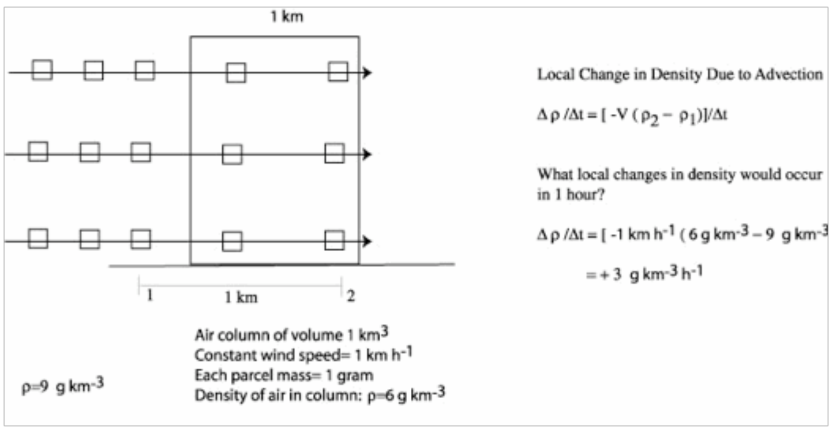


Figure 1: Conceptual illustration of changing the density (and mass) in an air column by differential import of higher density air replacing lower density air.

On a level surface weather map, lines of constant density are called isopycnals or isopycnics. Notice that at any level the streamlines would be extending from isopypcnics with higher values of density to areas with isopycnics with lower values of density. Hence, this is just another way of viewing density advection.

If the figure is looking at the air column with east to the right

$$Density\ Advection = -u\frac{\Delta\rho}{\Delta x} \quad \text{or} \quad -V\frac{\Delta\rho}{\Delta s} \qquad (1)$$

where the far right of (1) is the expression in natural coordinates. The inset on the right of Fig. 1 shows a solution of the right hand side of (1) by finite differences at one of the levels shown.

It turns out that except in narrow regions around fronts and other types of discontinuities (such as outflow boundaries) density advection is very small. In fact, in most cases it can be neglected. But we'll retain it for the sake of the discussion.

If there are no variations in density along a level surface, clearly there can be no horizontal density advection. But there is still another way of changing the density in that air column: by removing air laterally (or vertically) differentially. Figure 2 shows an example of the lateral motion pattern called horizontal divergence changing the density in an air column.

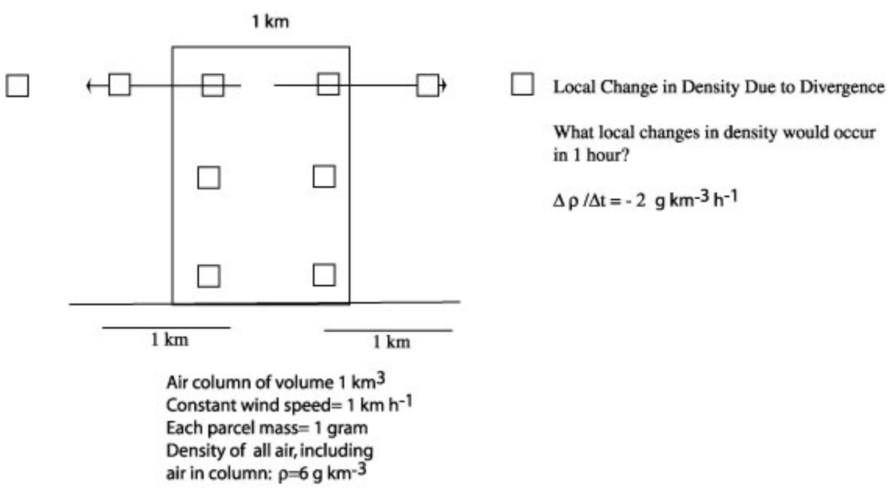


Figure 2: Conceptual illustration of changing the density (and the mass) in an air column by horizontal divergence of air streams. In this case, air parcels are removed from the top of the air column laterally.

Mathematically, the expression for this change in density due to horizontal or vertical air movement is shown here

$$Density\ Divergence = -\rho\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta w}{\Delta z}\right) = -\rho DIV \quad (2)$$

where

$$Divergence = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta w}{\Delta z} = DIV \quad (3)$$

where (2) is the expression for divergence in rectangular coordinates. The inset shows conceptually how divergence can decrease the density in the air column. Mathematically, divergence is positive when air parcels are spreading and negative when they are becoming closer together (called convergence).

The local changes in density are just the sum of Equations (1) and (2).

$$\frac{\Delta\rho}{\Delta t} = -\mathrm{V}\ \frac{\Delta\rho}{\Delta s} - \rho DIV \qquad (4)$$

which is one form of the Equation of Continuity. Away from boundaries with density differences (as described above) and for time intervals of days to seasons, the local changes in density are very small or negligible. Thus, in equation (4), the left and side and the first right hand term can be thought to be 0.

This results in the simplified synoptic-scale continuity equation, also known as Dine's Compensation.

$$\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} + \frac{\Delta w}{\Delta z} = 0 \qquad (5a)$$

or

$$\left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y}\right) = -\frac{\Delta w}{\Delta z} \qquad (5b)$$

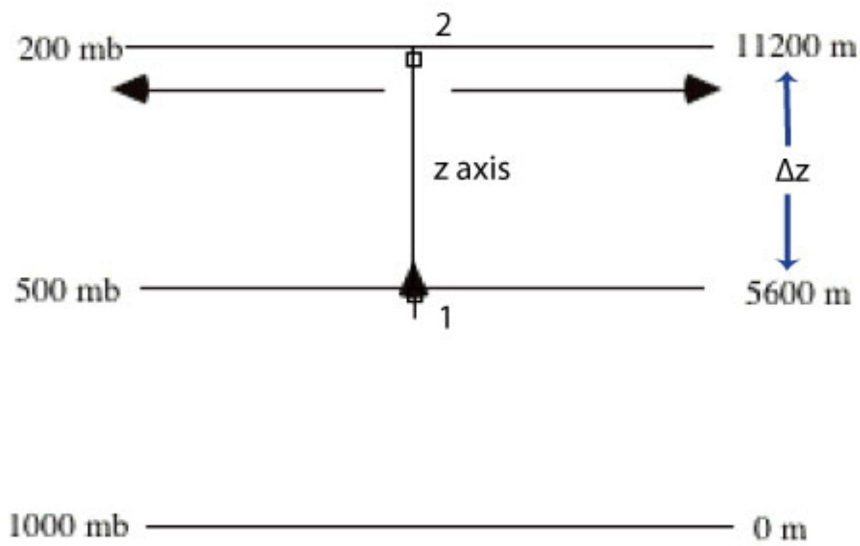
or

$$DIV_h = -\frac{\Delta w}{\Delta z} \qquad (5c)$$

$$DIV_h = -\left(\frac{w_2 - w_1}{z_2 - z_1}\right) \qquad (5d)$$

where (5d) is the finite difference form of (5c) in which the numbers refer to levels in the atmosphere. Equation (5c,d) is a very powerful expression that allows one to visualize how "forced" lofting (separate from buoyancy) occurs in slabs of our atmosphere. In other words, we have spent much time discussing how the w component of the wind can develop spontaneously due to instability. But the simplified Equation of Continuity (hereafter referred to as Dine's Compensation) can also be associated with w without buoyancy.

Inclass Example: On a given day, the horizontal divergence in the layer from 500 mb to the Tropopause at 200 mb is calculated to be 1.5 X 10^{-5} sec^{-1} You are asked to compute the vertical motion that would occur at 500 mb in "compensation" to this divergence. An important (helpful) constraint is that vertical motion is always zero at the tropopause and at the ground.



Drawing to help you conceptualize what is going on.

$$(DIV_h)_{UpperTrop} = -\left(\frac{w_2 - w_1}{\Delta z} \right) \qquad \text{(5c,d)}$$

Master Equation expressing Dine's Compensation

where $\Delta z = z_2 - z_1$

In this case, level 2 is the 200 mb level, level 1 is the 500 mb level. You want to solve for the vertical wind at the 500 mb level, which is at level 1.

$w_1 = DIV_h \, \Delta z + w_2$

but $w_2 = 0$, and $\Delta z = 11200\text{ m} - 5600\text{ m} = 5600\text{ m}$

so

$w_1 = 1.5 \times 10^{-5}\text{ s}^{-1} \times 5600\text{ m} = 8.4\text{ cm s}^{-1}$

The conceptual result of this is that Dine’s Compensation states that upper tropospheric divergence is associated with midtropospheric lofting (upward motion) and vice-versa. This allows meteorologists to relate divergence patterns in the upper troposphere to vertical motions that loft large groups of air parcels upward or downward in the mid troposphere.

If you followed what we did in this example, you'll understand that if you were given the 500 mb vertical motion of $w=8.4$ cm s^{-1} and asked to compute the DIVh in the layer whose bottom is 1000 mb you would get an answer of -1.5×10^{-5} sec^{-1}, which is convergence.

That is why, commonly, Dine's Compensation is often expressed this way: Upper tropospheric horizontal divergence tends to be balanced by lower level horizontal convergence (and vice versa) with compensating vertical motion in between.

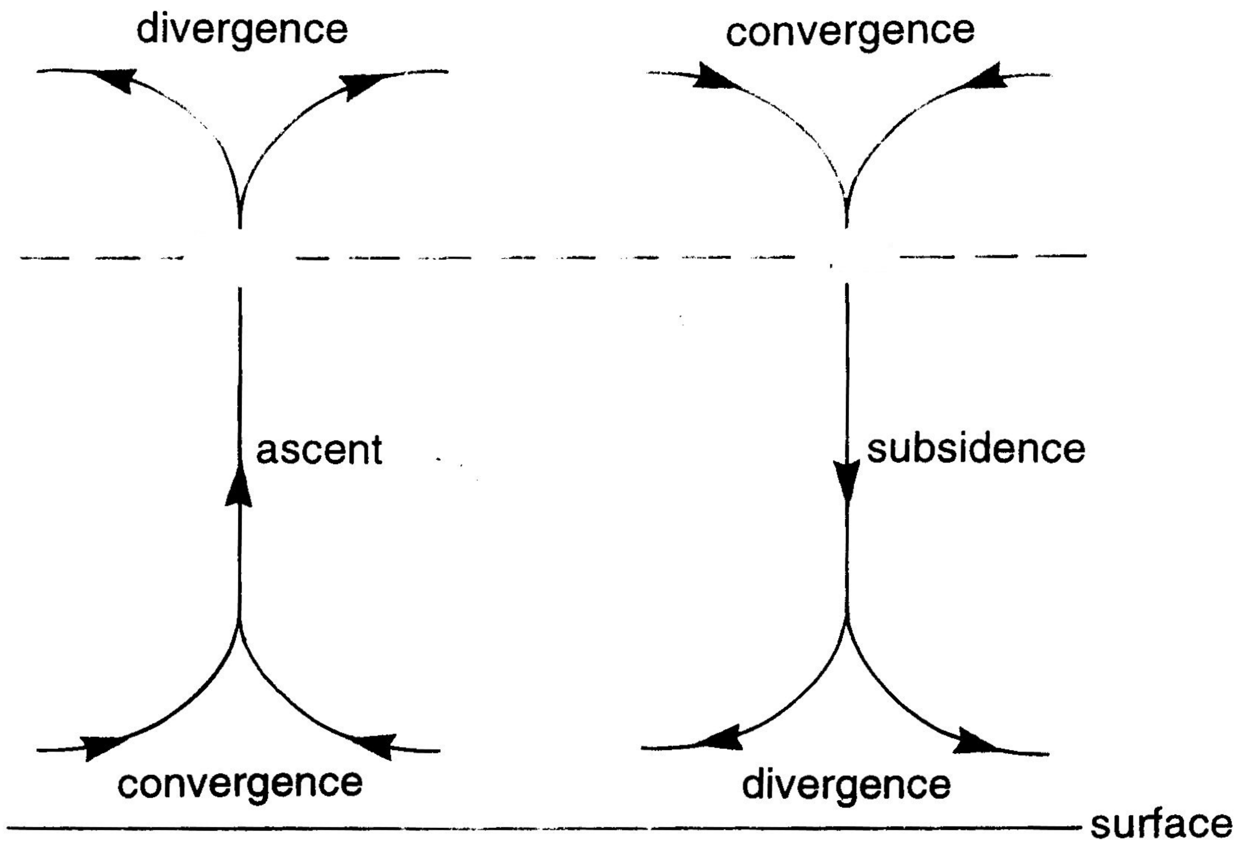


Figure 4: Visualization of Dine's Compensation. Technically, at the level of the dashed line, there is no horizontal divergence but only vertical motion. In the troposphere, that level tends to be in the midtroposphere, nominally 500 mb, and is often referred to as the Level of Non-divergence