

Mathematical Bases for Dry Adiabats and Saturation Mixing Ratio Lines: Poisson's Relation and the Clausius-Clapeyron Equation

A. Poisson's Relation

We conceptually developed your understanding of the dry adiabatic rate and dry adiabats in several Homeworks and In-class Exercises. But Poisson's Relation is the actual mathematical basis for the construction of (dry) adiabats on thermodynamic diagrams.

The equation is simply a combination of the Equation of State and the adiabatic term of the First Law of Thermodynamics.

The adiabatic term in the First Law of Thermodynamics is

$$\frac{\Delta T}{\Delta t} = \frac{1}{c_p \rho} \frac{\Delta p}{\Delta t} \quad (1)$$

The Ideal Gas Law is

$$p = \rho RT \quad (2)$$

where R is the gas constant for dry air. Solving for density in (2) and substituting into the right hand side of (1) gives

$$\frac{\Delta T}{\Delta t} = \frac{RT}{c_p p} \frac{\Delta p}{\Delta t} \quad (3a)$$

$$\frac{\Delta T}{T} = \frac{R}{c_p} \frac{\Delta p}{p} \quad (3b)$$

In calculus, you will learn how an expression like (3b) can be simplified. Also, $R/c_p = 0.286$.

Skipping this derivation until EARTH 465, equation (3b) becomes

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{0.286} \quad \text{or} \quad T_1 = T_2 \left(\frac{p_1}{p_2}\right)^{0.286} \quad (4)$$

where the subscripts 1 and 2 represent the values at a lower and higher level in our atmosphere. Equation (4) states that the air parcel's temperature, T_1 , at level 1 at which the pressure is p_1 can be related to the air parcel's temperature and pressure at level 2, where the temperature is T_2 and the pressure is p_2 .

If Level 1 is the 1000 mb level, then the temperature is the 1000 mb temperature. If Level 2 is the 500 mb level, then the temperature at level 2 is the temperature an air parcel would have if were brought dry adiabatically to a pressure elevation of 500 mb.

Equation (4) can be rewritten to solve for the temperature of an air parcel that ascends adiabatically from the 1000 mb level where the potential temperature is the 1000 mb temperature to a different isobaric level. But it is usually written in the form shown in (5)

$$\theta = T \left(\frac{1000 \text{ mb}}{p}\right)^{0.286} \quad (5)$$

which is Poisson's Relation where θ is the temperature (K) at 1000 mb, also known as the Potential Temperature.

Say the air parcel at 1000 mb has a temperature of 20C (293K) and we wish to know what its temperature would be if it were lofted to a pressure elevation of 500 mb.

$$293 = T(2)^{0.286}$$

$$293\text{K} = 1.2195 T$$

$$T = 240\text{K} = -33\text{C}$$

Poisson's Relation can be solved at other pressure elevations, for each 1000 mb temperature. The family of lines thus generated can be plotted on a diagram that has temperature (in Kelvin) as one coordinate, and p^k (increasing downward) as the vertical coordinate. If one does this for all possible temperatures and pressures, one obtains one form of a thermodynamic diagram called a [Stüve or Pseudoadiabatic Chart](#).

An example is provided below. For operational purposes, only the portion of the chart that has realistic temperatures for earth (1000 mb temperatures between 220 and 345) is generally reproduced. The lines are nearly parallel on such a small portion of the chart and are really potential temperature contours (also known as ISENTROPES and DRY ADIABATS).

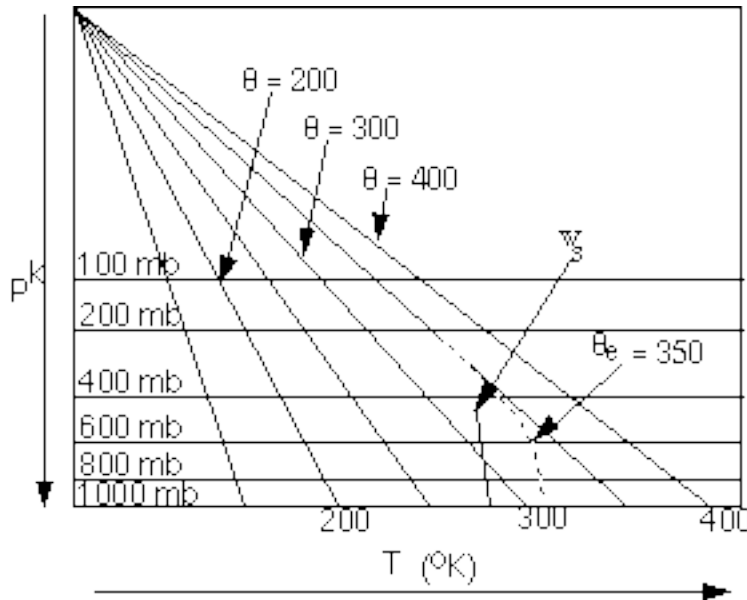


Figure 1. A Stüve diagram showing the solution of Poisson's Relation as a family of lines corresponding to the adiabatic rate, known as dry adiabats, or lines of constant potential temperature, known as isentropes. Dry adiabats/isentropes are often labeled with the value of potential temperature either in C or K.

B. Clausius-Clapeyron Equation

We developed a conceptual understanding of the saturation mixing ratio, saturation vapor pressure, and relative humidity empirically, by proceeding through the virtual fish tank experiment on evaporation. The mathematical basis for this is the Clausius-Clapeyron Equation. Simply stated, the Clausius-Clapeyron Equation relates the partial pressure of water vapor at saturation to the temperature.¹

$$e_s = (6.1904 \text{ kPa}) \left[\left(\frac{17.625T}{T+243.04C} \right) \right] \quad (6)$$

where T is in C and pressure is in kPa (1 kPa = 10 mb).

¹ An operational version of this is known as the **August-Roche-Magnus** approximation.

Equation (6) can be written in simplified form as follows

$$e_s = k^{[\sim T]}$$

The interesting thing about the exponent in this equation is that it essentially implies that the saturation vapor pressure directly dependent upon temperature. The higher the temperature (appearing as the T in the exponent), the larger the saturation vapor pressure.

For example, for an air temperature of 30C, Equation (6) gives:

$$e_s = (6.1904 \text{ kPa})^{[0.808]} = 4.36 \text{ kPa}$$

Since 1 kPa is 10 mb, 4.36 kPa is 43.6 mb. So, at 30C, the saturation vapor pressure is about 4.3% of the total average atmospheric pressure.

But saturation vapor pressure is not usually put on thermodynamic diagrams like the Skew T/log P. Equation (5) can be converted to saturation mixing ratio by the expression

$$W_s = \frac{(622 \text{ g kg}^{-1})e_s}{P - e_s} \quad (7)$$

For the example given here of $e_s = 4.36 \text{ kPa}$ and a pressure, say, of 1000 mb (100 kPa), equation (6) gives the following saturation mixing ratio for a temperature of 30C and a pressure of 1000 mb:

$$W_s = \frac{(622 \text{ g kg}^{-1})4.36 \text{ kPa}}{100 \text{ kPa} - 4.36 \text{ kPa}} = \frac{2711.92 \text{ g kg}^{-1}}{95.64} = 28.4 \text{ g kg}^{-1}$$

Equations (6) and (7) can be solved for every temperature and pressure conceivable in our atmosphere so that lines of constant saturation mixing ratio (isohumes) can be drawn. The result is as shown in Fig. 4.7 of Stull, given as Figure 2 here. On any thermodynamic diagram, the actual plotted temperature corresponds to the saturation mixing ratio shown at that point and the plotted dew point temperature the mixing ratio at that point. These two in combination can be used to calculate relative humidity.

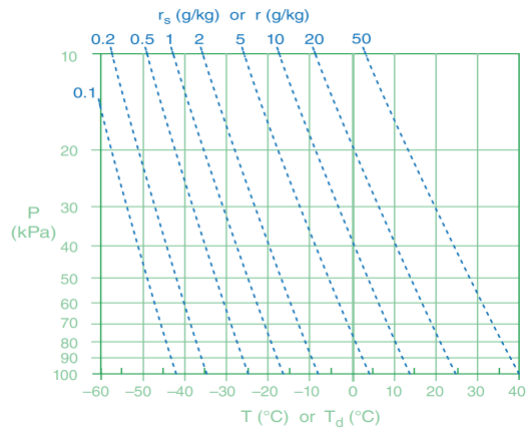


Figure 2: Lines of constant saturation mixing ratio shown on a Stüve Diagram.