The Continuity Equation and Dines Compensation

1. Review of ERTH 260 Continuity Equation Discussion

The Continuity Equation is a restatement of the principle of Conservation of Mass applied to the atmosphere. The principle simply states that matter can neither be created or destroyed and implies for the atmosphere that its mass may be redistributed but can never be "disappeared".

As we discussed in ERTH 260, the Equation of Continuity restates this by telling us that there are TWO basic ways to change the amount of air (density) within fixed volume of air (an air column, for the sake of this discussion, fixed with respect to the earth and extending from the ground to the top of the atmosphere): (i) if air is flowing laterally through the air column, have the upstream air more (less) dense than the downstream air, hence, air of different density replaces the air in the air column (density advection[1]) and (ii) even if density is everywhere constant, remove air from the air column of fixed volume (called three dimensional divergence) (Fig. 2).

Thus, the Equation of Continuity can be written

$$\frac{\partial \rho}{\partial t} = -\vec{V} \cdot \nabla \rho - \rho \nabla \cdot \vec{V} \qquad (1)$$

where $\vec{V}$ is the three-dimensional wind vector in rectangular coordinates and $\nabla \cdot \vec{V}$ is the three-dimensional divergence of the wind

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad (2)$$

The equation states that the rate of change of density observed at a fixed location is due partially to $-\vec{V} \cdot \nabla \rho$, which is the three-dimensional density advection and partially due to a three-dimensional divergence term, $-\rho \nabla \cdot \vec{V}$. As discussed in ERTH 260, the density advection acts to increase density locally if it is positive (positive density advection) and the divergence term acts to decrease density locally by net export of air out of the air column.

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[1]Advection is defined as the transport of an atmospheric property only by the velocity field (i.e., temperature advection, moisture advection etc.) and in scalar form is given as the product of the wind velocity component and the gradient of the property along the respective coordinate axis (e.g., -u ($\Delta T/\Delta x$)).

2. Synoptic Scaling: Dines Compensation

In this class, we are concentrating on the larger "scales" of atmospheric phenomena--the Macroscale (10000 km or so) and the Synoptic Scale (1000 km or so). At these scales, the density advection is negligible compared to the Velocity Divergence term. Also, at these scales, changes in density within an air column fixed with respect to the earth are also very small. [2]

At the synoptic scale, then, Equation (1) reduces to

$$\frac{\partial \rho}{\partial t} = 0 = -\rho \nabla \cdot \vec{V} \qquad (3a)$$

or

$$\nabla \cdot \vec{V} = 0 \qquad (3b)$$

By substitution of Equation (2), Equation (3b) can be rewritten

$$\left(\nabla \cdot \vec{V}\right)_h = DIV_h - \frac{\partial w}{\partial z} \qquad (4a)$$

or

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = DIV_h = -\frac{\partial w}{\partial z} \qquad (4b)$$

Equation (4b) can be written in finite difference form as follows

$$DIV_h = -\frac{\partial w}{\partial z} \approx -\frac{\Delta w}{\Delta z} = -\left(\frac{w_2 - w_1}{z_2 - z_1}\right) \qquad (5)$$

Equation (5) is the basis for the principle we learned in ERTH 260 is known as DINES COMPENSATION. The importance of the equation may be more

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[2] Both density advection and local changes in density experienced locally are NOT negligible at smaller scales. For example, the circulations near fronts (mesoscale) and outflow boundaries (thunderstorms) are strongly influenced by advection of density.

apparent now. It implies that we can say something about the field of vertical motion if we know something about the net horizontal divergence.

First, the units of divergence can be visualized as (m s^{-1})/m = s^{-1}. Remember that horizontal divergence is a measure of the percentage increase or decrease in the horizontal area of an air column in a unit amount of time, expressed as a ratio. Thus, if an air column increases its area by 20% in 1 day, then the horizontal divergence would be 1.2 dy^{-1}. The conventional unit is s^{-1}.

Now, observations show that the vertical velocity at the Tropopause and at the ground is nearly **zero.** Take a look at Figure 1, which shows a schematic cross section with qualitative wind vectors. Say that the horizontal divergence implied by this schematic in the upper troposphere is the horizontal divergence for the upper tropospheric layer (above the Level of Nondivergence). And let's say that this horizontal divergence is numerically equal to 1.5×10^{-5} s^{-1} (which is a typical value for synoptic scale horizontal divergence in the upper troposphere).

Let's use Equation (5) to say something about the midtropospheric vertical velocity field given these conditions.

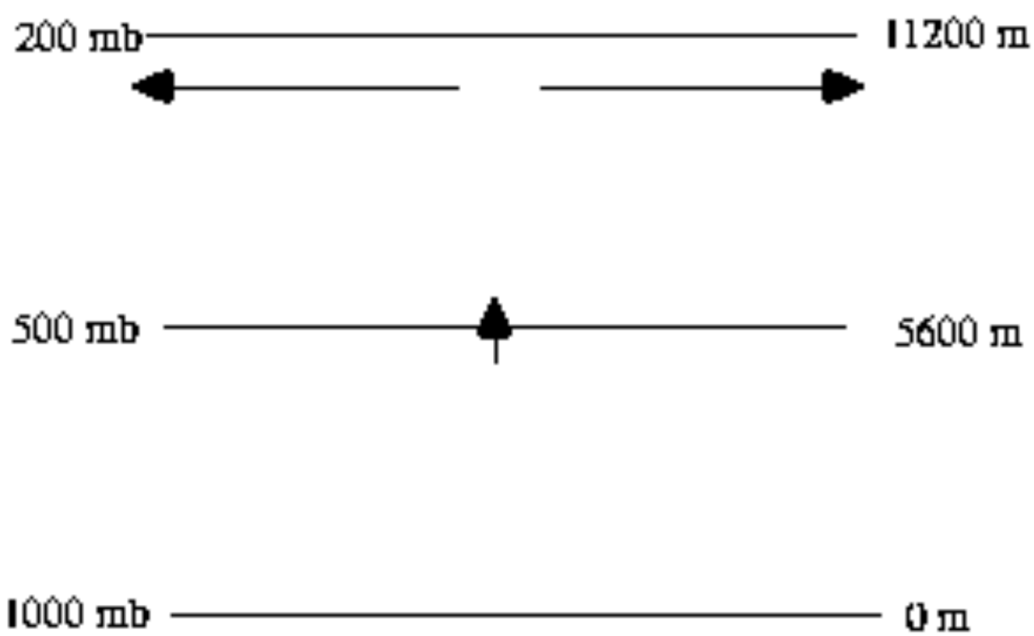

Figure 1: Schematic cross-section of the atmosphere showing qualitative wind vectors in the upper troposphere, and at the Level of Non-Divergence, as explained above.

You need to rewrite equation (5) to solve for the vertical velocity at 500 mb. To do this you need to expand the term to right of the equals sign, which is really the vertical divergence.

$$DIV_h = \approx -\left(\frac{w_2 - w_1}{z_2 - z_1}\right) = -\left(\frac{w_{200} - w_{500}}{z_{200} - z_{500}}\right) \qquad (6)$$

and solve for w_{500}

$$w_{500} = DIV_h(z_{200} - z_{500}) + w_{200} \qquad (7)$$

The constraints of the problem are

$w_{200} = 0$, $DIV_h = 1.5 \text{ X } 10^{-5} \text{ s}^{-1}$

$$w_{500} = 1.5 \text{ X } 10^{-5} \text{ s}^{-1}(11200m - 5600m) + 0$$
$$w_{500} = 0.084 \text{ m s}^{-1} = 8.4 \text{ cm s}^{-1}$$

Answer Check: Given the concept of Dines Compensation, we would expect upward vertical velocity at 500 mb, given the contraints of the problem. In addition, strong synoptic scale vertical motions are on the order of 10 cm/ s or so. The answer seems reasonable.