Fundamental Definition of Horizontal Divergence

In meteorological applications, loosely stated, horizontal divergence is a measure of the change in cross-sectional area of an air column. This change is divided by the original area of the air column to obtain a fractional rate of change. Thus, algebraically it can be written $\Delta A/A$, where A is the cross-sectional area of the air column. Dividing by the time over which this change is observed yields the fractional rate of change per unit time.

$$(\Delta A/A)/\Delta t \qquad (1a)$$

or

$$1/A\ (\Delta A/\Delta t) \qquad (1b)$$

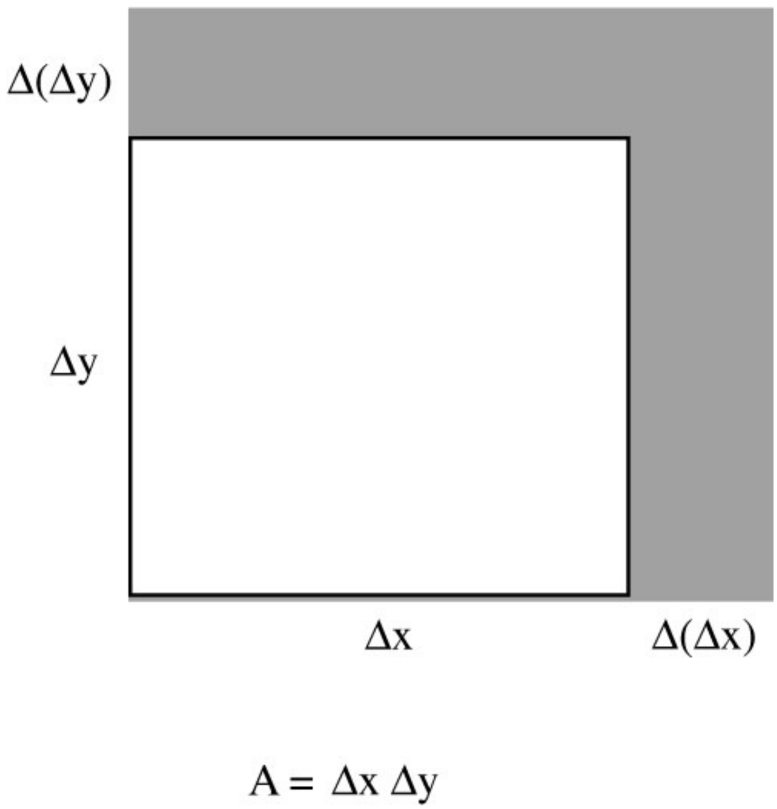
Horizontal divergence is defined as the fractional rate of change of cross-sectional area of an air column (ratio of the area change in a unit time to the original area), following the motion of the air column. Since we are following a moving air column and measuring changes relative to it (rather than relative to the surface of the earth), Equation (1b) can be converted to calculus notation, giving the fundamental definition of horizontal divergence[1].

$$DIV_H = \frac{1}{A}\left({dA}/{dt}\right) \qquad (1c)$$

Horizontal Divergence in Rectangular Coordinates

Consider an air column with a rectangular-shaped cross-sectional area that experiences a change in time (Fig. 1).

[1] In reality, divergence is three dimensional, but our discussions here center on what you can see on a horizontal map view.



The dimensions of the original air column are Δx and Δy.

$$A = \Delta x \Delta y \tag{2}$$

The finite difference version of **dA/dt,** after substituting (2), is

$$dA/dt \approx \Delta A/\Delta t = \frac{\Delta(\Delta x \Delta y)}{\Delta t} \tag{3}$$

Use the chain rule to expand right hand term.

$$\Delta A/\Delta t = \Delta y\left[\Delta\left(\Delta x/\Delta t\right)\right] + \Delta x\left[\Delta\left(\Delta y/\Delta t\right)\right] \tag{4}$$

but $\Delta x/\Delta t = u$ and $\Delta y/\Delta t = v$. Substituted into equation (4), this gives

$$\Delta A/\Delta t = \Delta y[\Delta u] + \Delta x[\Delta v] \qquad (5)$$

To obtain DIV_h, divide both sides of (5) by the area, $\Delta x\ \Delta y$

$$\frac{1}{A}\Delta A/\Delta t = \frac{\Delta y[\Delta u]}{\Delta x \Delta y} + \frac{\Delta x[\Delta v]}{\Delta x \Delta y} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \qquad (6a)$$

and take the limit as Δt->0, and substitute (5) into (1c) and remembering that u and v are defined by streamlines showing motion relative to the earth and can be estimated by the gradient in wind observations relative to the earth, equation (6a) can be written

$$DIV_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \qquad (6b)$$

This result is the same, as expected, as the operation

$$\nabla_h \bullet \vec{V} = \left(\vec{i}\ \partial/\partial x + \vec{j}\ \partial/\partial y\right) \bullet \left(u\vec{i} + v\vec{j} + w\vec{k}\right) \qquad (7)$$

Horizontal Divergence in Natural Coordinates

As we have found out, it is most often instructive and logical to consider kinematic quantities in the natural coordinate system.

What is the horizontal cross-sectional area of the air column in the natural coordinate system? Hence the fundamental definition of Divergence can be expanded in the same manner.

$$\nabla_h \bullet \mathbf{V} = \frac{1}{\mathbf{A}} \mathbf{dA} / \mathbf{dt} \approx \frac{1}{\Delta \mathbf{s} \Delta \mathbf{n}} \left[\Delta(\Delta \mathbf{s} \Delta \mathbf{n}) / \Delta \mathbf{t} \right]$$

$$\Delta s \ \Delta n = A$$

Equations (8a and 8b)

The term in brackets can also be expanded using the Chain Rule.

We note the following

$$\mathbf{V} = \Delta \mathbf{s} / \Delta \mathbf{t} \qquad (9)$$

The additional complication for this derivation in the natural coordinate system is that the streamlines often spread out or get closer together following the direction of the flow. To visualize this, examine Fig. 1.

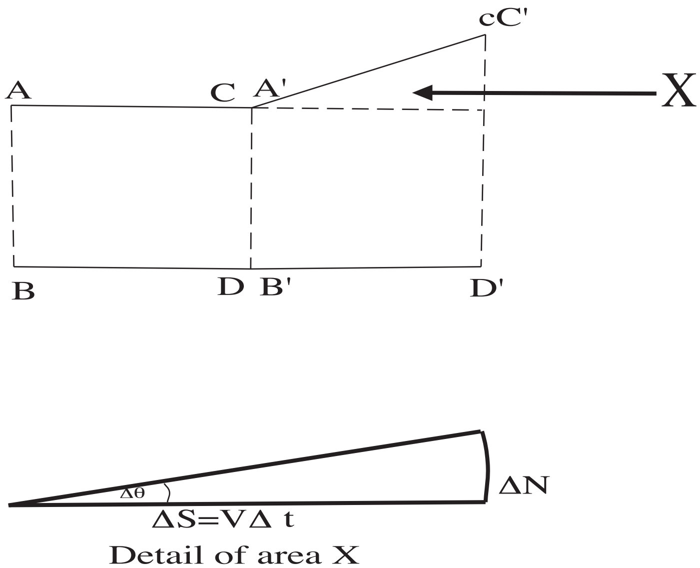


Figure 1: Schematic diagram of two streamlines that are initially parallel, but then spread out downstream, resulting in a change in cross sectional area of the initial air column.

The growth of ABCD to A'B'C'D' is proportional to the original area plus the area of the pie-wedged shape labeled with the arrow. This pie-shaped area has dimensions in the s and n directions as well as shown by the inset.

Since ΔS can always be related to the speed V by the relation $V = \Delta S/\Delta t$, the dimensions of the side ΔS is as shown in the inset, where V is constant during the time interval considered here.

The distance ΔN is an arclength that can be obtained as the product of the angle subtended and the distance ΔS. From the figure above, we therefore get:

$$\Delta n = \Delta\theta\Delta s \quad (10a)$$

$$\Delta s = V\Delta t \quad (10b)$$

Substitute (9), (10a) and (10b) into (8a) and converting to calculus notation, one gets

$$DIV_h = \frac{1}{A}\frac{dA}{dt} = V\left[\frac{\partial\theta}{\partial n}\right] + \frac{\partial V}{\partial s} \qquad (12)$$

which is the fundamental definition of divergence in natural coordinates. The first right hand term is called “diffluence” and the second is called “speed divergence”[2].

[2] Negative diffluence is called confluence and negative speed divergence is called confluence.