Equation of Motion: Cartesian and Natural Coordinate Systems

The Equation of Motion in rectangular coordinates, following our discussions of the last week or so, is:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi + \text{ Frictional Acceleration}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega u \sin\phi + \text{ Frictional Acceleration}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

(1a, b, c)

We can make the following simplifications for the SYNOPTIC SCALE free atmosphere:

1.   Frictional acceleration can be neglected above the boundary layer (above ~300 meters or so above the local ground level;

2.   In the free atmosphere, horizontally moving air parcels tend to be unaccelerated, except in restrictive circumstances, and only at certain times. The result of this is that the left hand side of (1a, b) tends to be one order of magnitude or more smaller than the magnitudes of the other terms. Note that this assumption is not true for strongly curved flow (discussed another time) and in the upper troposphere.

3.   Vertical accelerations are small compared to horizontal accelerations and vertical velocities are small (generally, one to two orders of magnitude smaller) compared to horizontal accelerations and velocities. At the synoptic scale, vertical velocities and accelerations are one to two orders of magnitude smaller than the the other forces per unit mass that affect air parcels.

Simplifications (1) and (2) transform (1a, b) to:

$$u_g = \frac{1}{\rho f} \frac{\partial p}{\partial y}$$

$$v_g = -\frac{1}{\rho f} \frac{\partial p}{\partial x}$$

(2a, b)

the component geostrophic wind equations, which state that the speed of the wind is directly proportional to the pressure gradient, and parallel to the isobars (and non-divergent, which you proved in Lab 2).

And simplification (3) transform (1c) to:

$$\frac{\partial p}{\partial z} = -\rho g$$

(2c)

the hydrostatic equation, which states that, with the simplifications accepted, the vertical pressure gradient acceleration is balanced by the acceleration of gravity.

The problem with Equations (2a,b) is that there is no direct way to measure density. However, density is related to the vertical pressure gradient at the synoptic scale, as shown in (2c). Thus, we can transform equations (2a,b) from the rectangular coordinate system to the x, y, p coordinate system by in the procedure we outlined in the previous handout, and perform the substitutions into (2a,b) to produce the geostrophic wind components in the for the x, y, p coordinate system.

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y}$$

$$v_g = \frac{g}{f}\frac{\partial z}{\partial x} \qquad\qquad\qquad (3a, b)$$

where z is the geopotential height (and the derivatives measure the variation of geopotential height along the horizontal coordinate axes).

Equation of Horizontal Motion in Natural Coordinates

The Equation of Horizontal Motion can be transformed into natural coordinates. The equation can be stratified on the basis of accelerations that act tangent to the flow (t component) and accelerations that act normal to the flow (n component). But by definition a horizontal acceleration that affects, say, a stationary air parcel will produce a motion in the positive direction along the streamline.

One additional term that appears explicitly relates to the curvature of the streamlines ONCE the air is in motion. Air involved in straight flow in geostrophic balance that encounters curved isobars will at first continue to move in a straight line relative to the isobars. Relative to the earth, this creates a local "centrifugal acceleration" that acts normal to the flow, tending to throw the air outward relative to the curved isobars. This interesting effect creates what is known as the "Gradient Wind" (to be discussed in future classes).

You know from your physics classes that the expression for centrifugal acceleration per unit mass is

$$\frac{V^2}{r} = \text{Centrifugal Acceleration per Unit Mass}$$

where r is the radius of curvature (distance to the local center of curvature). Fluid dynamicists and meteorologists define a quantity called curvature that is defined as 1/r=K.

Because centrifugal acceleration acts normal to the flow, in natural coordinates, it appears as an acceleration that acts in the n direction. Thus the equation of horizontal motion in natural coordinates may be written as follows, stratified by those forces per unit mas (the accelerations) that act in the t and n directions:

$$\frac{dV}{dt}\vec{t} + \left(KV^2\right)\vec{n} = -\left(\frac{1}{\rho}\frac{\partial p}{\partial s}\right)\vec{t} - \left(\frac{1}{\rho}\frac{\partial p}{\partial n} + fV\right)\vec{n} \quad (5)$$

Full Equation of Motion in Natural Coordinates

$$\frac{dV}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial s} \quad (6)$$

t components in Equation (5)

$$KV^2 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV \quad (7)$$

n components in Equation (5)

Balanced Flow: Flow Along Contours

1. What about balanced flow along straight contours? <u>The Geostrophic Wind</u>

The wind flow is termed geostrophic when the wind vector is tangent to the isobars (hence, there is no pressure difference along the streamlines).   If there is no pressure gradient along the streamlines, then there will be no real accelerations (meaning, the total derivative of the velocity is zero...the air parcels speed won't change).   Hence, both t components are zero.

$$ \frac{dV}{dt} = - \frac{1}{\rho} \frac{\partial p}{\partial s} = 0 $$

The wind flow is termed geostrophic for an invarying pressure gradient normal to **straight** flow. Hence, there is no local centrifugal acceleration. Solving for V (now called the geostrophic wind)

$$ V_g = - \frac{1}{\rho f} \frac{\partial p}{\partial n} = - \frac{g}{f} \frac{\partial z}{\partial n} \qquad \text{(8)} $$

These are the geostrophic wind equations (in natural coordinates) in the s, n, z and s, n, p coordinate systems respectively. The far right hand equation is used mostly in describing the geostrophic wind conceptually.

2. What about balanced flow for curved contours? <u>The Gradient Wind</u>

For streamlines plotted relative to the earth's surface, a local centrifugal acceleration will exist for curved isobars/streamlines. Air initially in geostrophic balance moving from a region of straight to a region of curved contours will experience a centrifugal acceleration relative to the center of curvature.

Initially, since the air is moving tangent to the isobars, the t components (shown in Equation (6)) are still zero.   However, there is a local centrifugal acceleration proportional to the curvature of the air parcel's trajectory that tends to throw the air parcel across the isobars or height contours)

First divide both sides of the n-components (Equation (7)) by f.

$$\frac{1}{f} K V^2 = - \frac{1}{\rho f} \frac{\partial p}{\partial n} - V$$

(9)

now we'll solve for V, except now we'll call this the gradient wind and place a subscript to indicate that, V_{gr}.

$$V_{gr} = - \frac{1}{\rho f} \frac{\partial p}{\partial n} - \frac{1}{f} K V^2$$

(10)

At a given latitude, the far right hand term is simply proportional to the local centrifugal acceleration (a vector that tends to "throw" air parcels normal to curved contours). Substituting the geostrophic wind equation (equation (8) above into (4) and rewriting gives

$V_{gr} = V_g$ +/- Velocity Due to Local Centrifugal
Acceleration

The +/- arises in the second term due to the convention that cyclonic curvature is positive and anticyclonic curvature is negative.

For cyclonically curved contours, the far right term acts in a direction in the negative "n" direction, opposite the pressure gradient acceleration (and, thus, subtracts from it) and for anticyclonically curved contours, the far right term acts in the positive "n" direction, the same direction as the pressure gradient acceleration (and, thus, adds to it). Thus, around troughs, the gradient wind is subgeostrophic (meaning, the wind speed is less than what would be predicted by the geostrophic wind equation) and vice versa around ridges.

If the contours are straight, then the far right term is zero, and the equation collapses to the geostrophic wind relation. We'll see that in the mid-troposphere, the gradient wind is also nearly non-divergent, just as the geostrophic wind is non-divergent.

3. Manipulate the expressions above to create an "Equation of Motion" for the state of zero flow (calm). This allows one to understand the basic reason for air motion.

In a situation of calm, but with a pressure pattern having developed on a weather map, initially there is no velocity. Hence two of n-components immediately are zero

However, equation (1a) says air motion WILL develop if there is a pressure gradient acceleration...and that acceleration will act to take the air parcel at right angles to the isobars or height contours. The air will begin moving directly across the isobars towards lower pressure, with the s-axis pointing directly towards the low pressure area (at right angles to the isobars). The n-axis will be tangent to the isobars, and hence, there still will be no pressure gradient normal to the flow. Thus, all terms of equation (1b) are zero, and the equation of motion is completely encompassed by equation (1a).

4. What about curved and balanced flow at very small distance scales (low end mesoscale and microscale): the Cyclostrophic Wind

Here are the components of the Equation of Motion, restated:

$$\frac{dV}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial s}$$

(6)

t components in Equation (5)

$$KV^2 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV$$

(7)

n components in Equation (5)

Flow around tornadoes tends to be parallel to local isobars or height contours, and the same is true for flow around and in the eyewalls of hurricanes. For the very short distances around tornadoes and short distances around the center of hurricanes, Coriolis Effect (and accelerations) are one to three orders of magnitude smaller than the local centrifugal accelerations at the radius of curvature of each (~100 m for tornadoes, ~50 km for hurricanes). The pressure gradient acceleration is also two to three orders of magnitude larger than the Coriolis Acceleration, typically.

In Homework #5, you will simplify equations (6) and (7) (and, thus, (5) by consideration of these factors.

The resulting equation of motion is for balanced flow, with the resulting wind called the cyclostrophic wind.

$$V_{cyclostrophic} = \sqrt{-\frac{1}{K}\left(\frac{1}{\rho}\frac{\partial p}{\partial n}\right)} = \sqrt{-\frac{r}{\rho}\frac{\partial p}{\partial n}}$$

$$V_{cyclostrophic} = \sqrt{-r\, g\, \frac{\partial z}{\partial n}} \qquad\qquad (11a,b)$$

where (11b) is the cyclostrophic wind in isobaric (x, y, p) coordinates.