Lagrangian and Eulerian Derivatives and the Temperature Tendency Equation

A. The Total Derivative

The total derivative estimates the change of a dependent variable (say, temperature) in, say, an air parcel no matter where it is and no matter whether it is moving or not. This air parcel can be visualized as having a coordinate system fixed with respect to it.

But that change can be related to the partial contributions of the changes that occur with respect to time and space measured relative to coordinate systems fixed with respect to the surface of the earth.

The total change of any variable, say temperature, can occur in time and in space. Since the Cartesian coordinate system has three axes, this can be expressed algebraically by (1a) and in calculus notation by (1b). Usually, in calculus text books the general symbol f is used for the dependent variable.

$$\Delta T = \left(\frac{\Delta T}{\Delta t}\right)_{Fixed}\Delta t + \left(\frac{\Delta T}{\Delta x}\right)_{Fixed}\Delta x + \left(\frac{\Delta T}{\Delta y}\right)_{Fixed}\Delta y + \left(\frac{\Delta T}{\Delta z}\right)_{Fixed}\Delta z$$

$$\mathrm{d}T = \frac{\partial T}{\partial t}\mathrm{d}t + \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial y}\mathrm{d}y + \frac{\partial T}{\partial z}\mathrm{d}z \qquad \text{(1a, b)}$$

Equation (1) states, conceptually, that the total change in temperature, T, is directly related to the rate of change of T in time and space. You can see how the calculus notation makes the equation easier to work with.

B. The Lagrangian and Eulerian Derivatives

To make an expression more useful to meteorologists (and oceanographers), one can use the rules of calculus and divide both sides of (1) by the differential dt to obtain the so-called "total derivative" of *T*. Note that now the **partial** derivatives also have an interpretation of "as measured with respect to a coordinate system fixed with respect to the surface of the earth" while the conventional derivative notation indicates measure as if the coordinate system moves with the moving object, in this case, an air parcel.

$$\frac{\mathrm{dT}}{\mathrm{d}t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \qquad (2)$$

But, in ERTH 260 (Metr 201) we learned that the definition of the wind components in the cartesian coordinate system is:

$$\mathrm{u} = \frac{\mathrm{dx}}{\mathrm{dt}} \approx \frac{\Delta \mathrm{x}}{\Delta \mathrm{t}},\ \mathrm{v} = \frac{\mathrm{dy}}{\mathrm{dt}} \approx \frac{\Delta \mathrm{y}}{\Delta \mathrm{t}},\ \mathrm{w} = \frac{\mathrm{dz}}{\mathrm{dt}} \approx \frac{\Delta \mathrm{z}}{\Delta \mathrm{t}} \qquad (3)$$

where u is the west (negative u is east) wind, v is the south (negative v is north) wind and z is the up (negative z is down) wind component.

Substitute (3) into (2) and we have the equation for the so-called Lagrangian (or material) Derivative[1] form of the total derivative, where (4b) is the expression in vector calculus notation.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

or

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \bullet \nabla T \qquad (4a, b)$$

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[1] Please note that in some books the Lagrangian Derivative is still referred to as the total derivative and will appear, even in some meteorology texts, as dT/dt.

$\frac{DT}{Dt}$ is known as the Lagrangian derivative

Since we are applying the concept of the total derivative to fluid motion, we use the capital D to indicate special application of the total derivative to fluid motion. It is interpreted as the change of temperature, in this case, measured following the motion of the parcel.

$\frac{\partial T}{\partial t}$ is known as the local or Eulerian derivative

and is interpreted as the change in temperature measured at a fixed location, for example, San Francisco Airport.

$$+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=+\vec{V}\bullet\nabla T$$

is known as temperature advection (usually brought to the other side of the equation so it will have a negative sign. This measures the contribution to local temperature change that occurs when air of different temperatures is brought into an area.

C. The Simplified Temperature Tendency Equation

Commonly, in weather forecasting, equation (4) is used generally with ANY variable that can characterize the state of an air parcel (for example, temperature, pressure, mixing ratio, humidity etc.) substituted for "f" and is solved for the local derivative. This yields an equation that allows us to forecast f at a fixed location.

$$\frac{\partial f}{\partial t} = \frac{Df}{Dt} - \vec{V} \bullet \nabla f \qquad (5)$$

$-\vec{V} \bullet \nabla f$ is known as the advective derivative (or just as advection). It measures the contribution to the change in f observed locally because of fluid with a different value of f (than the initial value) being brought to the observing site.

Any variable can be substituted for f, as we have already done so, for temperature or T.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

or using vector calculus notation

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \bullet \nabla T \qquad (5a, b)$$

or solving for the local derivative.

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \vec{V} \bullet \nabla T \qquad (7)$$

Equation (7) is called the simplified Temperature Tendency equation. This equation allows one to make a deterministic temperature forecast for a location by multiplying both sides by the time increment over which the forecast is to be made.

Please note that the far right term of equation (7) represents the three dimensional temperature advection.

$$-\vec{V} \bullet \nabla T = \text{Three Dimensional Temperature Advection}$$

It relates the temperature observed by a thermometer at a fixed location to the changes in temperature in the air parcels themselves and the horizontal and vertical advection of parcels with differing temperatures to that location. In most applications, the vertical temperature advection term is very small and can be dropped out on order of magnitude basis.

When this term is evaluated, it can yield a positive result (temperature increases due to temperature advection known as 'warm advection') or a negative result (temperature decreases due to temperature advection are known as 'cold advection'). But the advection is calculated INCLUDING the algebraic negative sign at the front of the expression.